Fig. 9a is common to each case. The rms miss distance is given by the relative separation at 10 sec ($T_{GO} = 0$).

The results clearly demonstrate the applicability of CADET to the evaluation of terminal miss distance for an acceleration-limited missile. The CADET and Monte Carlo results agree to within 10%. In addition, it is interesting to note that the miss distance obtained with the linearized system model (Fig. 9b) is entirely misleading in the presence of missile lateral acceleration limitations. The fundamental reason for the breakdown of the linear analysis is the T_{GO}^{-1} dependency of the loop gain which causes a rapid increase in the commanded airframe acceleration near intercept. Guidance control is effectively lost when the airframe saturates, and the result is a marked increase in miss distance as compared to that obtained by use of linear system models.

Conclusions

CADET is a powerful new approach to the direct statistical analysis of nonlinear systems. It is applicable to high-order systems with multiple nonlinearities, multiple inputs, and non-gaussian statistics. Its advantages are simplicity and economy; a disadvantage is that it is not exact. Another disadvantage, common to all describing function approaches, is that no suitable accuracy analysis exists. It follows that some Monte Carlo trials should always be made to validate the performance of a CADET simulation. On balance, however, the advantages of CADET are thought to be sufficient to ensure its substantial application to the direct statistical analysis of nonlinear systems.

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Minimax Design of Kalman-Like Filters in the Presence of Large-Parameter Uncertainties

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The Kalman filter has been used in many applications; however, practical implementation of the filter has required exact knowledge of the various system parameters (input and measurement noise covariance) in order to yield optimum performance. This paper develops a minimax technique for the direct synthesis of Kalman-like estimators when there are large uncertainties in the a priori statistics of the plant and measurement noises. Both continuous and discrete estimators are considered. General properties of the filters that satisfy the various minimax performance indices are discussed and a number of examples of both continuous and discrete applications are then presented to demonstrate the technique.

Introduction

THE Kalman filter,[1,2] designed to estimate the states of a linear system driven by white noise using measurements corrupted by white noise, has been used in many applications. However, practical implementation of the filter has required specific knowledge of the various system statistical parameters (input and measurement noise covariance) in order to yield optimum performance. If there is significant uncertainty associated with these parameters, a sensitivity analysis is usually carried out to evaluate the actual filter performance. It is the intent of the work presented herein to guide the designer through a systematic procedure which will yield a filter that performs in an acceptable fashion when there is significant uncertainty in the various system parameters. Very general restrictions are placed upon these uncertainties, and one will find that they are applicable in most practical situations.

The basis for the results in this paper is contained in several papers by D'Appolito and Hutchinson,[3–6] where the idea of

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using a minimax performance index to design filters when there are uncertainties in the a priori information is formulated. This idea is based upon previous results from applying game theory to optimum control problems as in Dorato and Kestenbaum[7] and Salmon.[8] The design criterion involves finding the proper filter gain matrix which minimizes the maximum variance of some function of the actual error in the estimate. There are several forms of the minimax filter depending upon the particular performance criterion one chooses.

The Continuous Problem Formulation

Consider the linear plant

$$\dot{x}(t) = F(t)x(t)+u(t) \tag{1}$$

with noisy measurement

$$z(t) = H(t)x(t)+w(t); \quad t_0 \leqq t \leqq T \tag{2}$$

$x(t)$, $u(t)$, and $z(t)$ are column vectors of n, n, and m dimension, respectively. F and H are matrices of appropriate dimension. $x(t_0)$ is Gaussianly distributed with zero mean and covariance P_0, and $u(t)$ and $w(t)$ are uncorrelated zero-mean Gaussian white noise processes such that

$$\text{Cov}\,[u(t)] = E\{u(t)u^T(\tau)\} = Q(t)\delta(t-\tau); \quad Q \geqq 0 \tag{3}$$

and

$$\text{Cov}\,[w(t)] = E\{w(t)w^T(\tau)\} = R(t)\delta(t-\tau); \quad R > 0 \tag{4}$$

The uncertain matrices Q and R are assumed to lie in compact convex sets V_Q and V_R. For example

$$V_Q = \{Q(t)|Q(t) = Q^T(t), \quad Q(t) \geqq 0$$

and

$$0 \leqq q_{\min} \leqq tr[Q(t)] \leqq q_{\max} < \infty\} \tag{5}$$

is one such set. For convenience the set $V = V_Q \times V_R$ with elements v is defined.

Lacking exact knowledge of Q and R, a filter identical in form to the Kalman filter is chosen a priori to estimate $x(t)$. Now, however, the gain will be adjusted to satisfy an appropriate sensitivity criterion. Specifically, the filter takes the form

$$\dot{\bar{x}}(t) = F(t)\bar{x}(t)+K(t)[z(t)-H(t)\bar{x}(t)] \tag{6}$$

$K(t)$ is limited to a compact convex set V_K of sufficient domain to generate the Kalman filter for every $v \varepsilon V$. $\bar{x}(t)$ is an unbiased estimate of $x(t)$. Let

$$M(t) = E\{[x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]\}^T \tag{7}$$

Then for a given Q and R

$$\dot{M}(t) = (F-KH)M+M(F-KH)^T+Q+KRK^T; \quad M_0 = P_0 \tag{8}$$

The mean square error of filter (6) is $tr[M(t)]$. Let the filter performance index be

$$J_M(T) = tr[M(T)] \tag{9}$$

A more general form of $J_M(T)$ is found in Ref. 5 where $M(T)$ is replaced by $WM(T)$, W being an arbitrary weighting matrix. For a given Q and R the minimum value of $J_M(T)$ is[2]

$$J_0(T) = tr[P(T)] \tag{10}$$

where $P(t)$, the covariance of the optimal estimate, satisfies the matrix Riccati equation

$$\dot{P}(t) = F(t)P(t)+P(t)F^T(t)+Q(t)-P(t)H^T(t)R^{-1}H(t)P(t);$$
$$P(t_0) = P_0 \tag{11}$$

and the optimal filter gain is

$$K_0(t) = P(t)H(t)R^{-1}(t) \tag{12}$$

J_M is a function of the unknown matrices Q and R and the gain K. From the definition of J_0

$$J_M(K,v,P_0,t_0,T) \geqq J_0(v,P_0,t_0,T) \geqq 0; \quad v \varepsilon V \text{ and } K \varepsilon V_K \tag{13}$$

J_M is one performance measure for filter (6). It is also appropriate to measure the performance of filter (6) in terms of its absolute or relative departure from optimality. These performance measures take the form

$$S^A(T) = J_M(K,v,P_0,t_0,T)-J_0(v,P_0,t_0,T) \tag{14}$$

and

$$S^R(T) = \{[J_M(K,v,P_0,t_0,T)-J_0(v,P_0,t_0,T)]\}/[J_0(v,P_0,t_0,T)] \tag{15}$$

Since v is uncertain, it seems most appropriate to select K to satisfy one of the following criteria:

$$S_1(P_0,t_0,T) = \min_{K \varepsilon V_K} \max_{v \varepsilon V} J_M(K,v,P_0,t_0,T) \tag{16}$$

$$S_2(P_0,t_0,T) = \min_{K \varepsilon V_K} \max_{v \varepsilon V} S^A(K,v,P_0,t_0,T) \tag{17}$$

$$S_3(P_0,t_0,T) = \min_{K \varepsilon V_K} \max_{v \varepsilon V} S^R(K,v,P_0,t_0,T) \tag{18}$$

All three criteria are minimax in nature. The S_1 criterion places a least upper bound on J_M in the presence of uncertain parameters and may be considered a "worst case" design. The S_2 and S_3 criteria seek to control filter sensitivity directly by minimizing the maximum absolute or relative deviation of J_M from its optimum value over the uncertain parameter set. The dependence of J_0 and P_0 complicates matters since any uncertainty in P_0 must be included in the set V. Under fairly general conditions J_0 is independent of P_0 as $T \to \infty$. Assuming that F, H, Q, and R are constant, bounded in norm, and that the system defined by Eqs. (1) and (2) is uniformly completely controllable and observable, Kalman and Bucy[2] have shown that every solution of the variance equations, (11), starting at a symmetric non-negative matrix P_0 converges to a unique constant non-negative matrix $\bar{P}$ as $t \to \infty$. For constant plants, P and J_0 are functions of $T-t_0$ only. For convenience let $t_0 = 0$.

It is possible to show, by appropriate limiting arguments,[5] that a unique steady-state solution exists for S_1, S_2, and S_3 as $T \to \infty$. Denoting the steady-state solutions of Eq. (8) by $\bar{M}$, and defining

$$\bar{J}_M(K,v) = tr[\bar{M}] \tag{19}$$

and

$$\bar{J}_0(v) = tr[\bar{P}] \tag{20}$$

one can then define a simpler set of performance criteria

$$\bar{S}_1 = \min_{K \varepsilon V_K} \max_{v \varepsilon V} \bar{J}_M(k,v) \tag{21}$$

$$\bar{S}_2 = \min_{K \varepsilon V_K} \max_{v \varepsilon V} \bar{J}_M(K,v)-\bar{J}_0(v) \tag{22}$$

$$\bar{S}_3 = \min_{K \varepsilon V_K} \max_{v \varepsilon V} \{[\bar{J}_M(K,v)-\bar{J}_0(v)]/\bar{J}_0(v)\} \tag{23}$$

The Discrete Problem Formulation

Consider an nth-order linear and time-invariant dynamical system with discrete measurements described by the relations

$$\dot{x}(t) = Fx(t)+u(t) \tag{24}$$

$$z(t_n) = Hx(t_n)+w(t_n) \tag{25}$$

where $x(t)$ is the state of the plant, a vector with n components, $u(t)$ is the input (a noise process), a vector with n components, $z(t_n)$ is the output at the nth time instant, an m-vector ($m \leqq n$), and $w(t_n)$ is an m-vector of measurement errors at the nth instant. The matrices F and H are the system and output matrices, respectively. It is assumed that the system is uniformly completely controllable and observable. The input vector $u(t)$ is composed of n components which are assumed to be uncorrelated zero-mean Gaussian noise processes and the covariance matrix of $u(t)$ is given again by Eq. (3). The random sequence $\{w(t_n)\}$ is an uncorrelated zero-mean Gaussian noise sequence with covariance matrix

$$E\{w(t_i)w^T(t_j)\} = R\delta_{ij} \tag{26}$$

where δ_{ij} is the Kronecker delta. It is assumed that the uncertain parameters Q and R are contained in the set V as before.

Because the system considered is one whose output is only available at discrete instances of time, a discrete model of the system must be derived as a basis for any discrete filter which one intends to design. One must not lose sight of the fact, though, that the uncertainties in the input noise covariances originally arise in the continuous representation of the system, although they will easily carry over to the discrete model.

The equations describing the discrete model of the system are

$$x(t_n) = \Phi(t_n - t_{n-1})x(t_{n-1}) + \Gamma(t_n)$$
$$z(t_n) = Hx(t_n) + w(t_n) \tag{27}$$

or

$$x_n = \Phi(\Delta t)x_{n-1} + \Gamma_n$$
$$z_n = Hx_n + w_n \tag{28}$$

where

$$\Phi(\Delta t) = \Phi(t_n - t_{n-1}) \tag{29}$$

The random sequence $\{\Gamma_n\}$ is an uncorrelated zero-mean Gaussian noise sequence with covariance matrix

$$E\{\Gamma_i \Gamma_j^T\} = Q'\delta_{ij}$$

where

$$Q' = \int_{t_{n-1}}^{t_n} \Phi(\tau)Q\Phi^T(\tau)\,d\tau$$

For each $Q \varepsilon V_Q$, it is seen from above that Q' is uncertain. If Q is symmetric positive semidefinite and has finite trace, so does Q' and therefore for each Q, Q' is well defined.

If the uncertain parameters, Q and R, are known, then to obtain the optimum estimate of $x(t)$, one simply builds the Kalman filter based upon these parameters. Without exact knowledge of Q and R, a suitable choice for the structure of the filter would still be one identical to the Kalman filter. However, the choice of the gain K will again be dictated by a specified sensitivity criterion. The filter would be of the form

$$\bar{x}_n = \Phi(\Delta t)\bar{x}_{n-1} + K[y_n - H\Phi(\Delta t)\bar{x}_{n-1}] \tag{30}$$

where K is assumed to be contained in a compact convex set V_K. It is also assumed that for every $K \varepsilon V_K$ the filter is asymptotically stable and that for every $v \varepsilon V_Q \times V_R$ an appropriate gain is contained in V_K to generate the optimum Kalman filter.

Let M_n denote the covariance matrix of the actual error in the estimate after a measurement, i.e.,

$$M_n = E\{[x_n - \bar{x}_n][x_n - \bar{x}_n]^T\} \tag{31}$$

where M_n satisfies the recursive equation

$$M_n = (I - KH)\Phi M_{n-1}[(I - KH)\Phi]^T + (I - KH)Q'(I - KH)^T + KRK^T \tag{32}$$

The total mean square estimation error of the filter, Eq. (30), in steady state is

$$\bar{J}_M = tr[\bar{M}(Q, R, K)] \tag{33}$$

where $M_n = M_{n-1} = \bar{M}$ in Eq. (32). From Eq. (33) it is noted that $\bar{M}$ is a function of both the uncertain system parameters and the adjustable filter parameter K. In like manner for a given Q and R the minimum of J_M, J_0 is given by Eq. (20) where now $\bar{P}$ is obtained from the relations

$$\bar{P} = (I - K_0H)\Phi\bar{P}[(I - K_0H)\Phi]^T + (I - K_0H)Q'(I - K_0H)^T + K_0RK_0^T \tag{34}$$

$$K_0 = \bar{P}H^TR^{-1} \tag{35}$$

Because Q and R are uncertain and the choice of K is at the discretion of the designer, the performance criteria defined by Eqs. (21), (22), and (23) are appropriate here as well.

General Properties of Minimax Filters

A number of properties characterize both the minimax process and the resultant filters. These properties are presented here without proof. The properties apply equally well to both continuous and discrete filters. The proofs for the continuous case are due to D'Appolito[5] and for the discrete case to Bongiovanni.[9]

As stated in the previous section, the matrix K of Eq. (6) or (30) is assumed to lie in a compact convex set V_K which contains $\mathscr{K}_0$ where

$$\mathscr{K}_0 = \{K \mid K = K_0 \text{ for some } v \varepsilon V\} \tag{36}$$

The assumption that $\mathscr{K}_0 \varepsilon V_K$ is not unreasonable because for a stable filter, P in Eq. (11) or (34) is bounded, implying K_0 is bounded from Eq. (12) or (35). Therefore a V_K does exist which contains $\mathscr{K}_0$.

The properties of the S_1 filter for constant but uncertain Q and R are now presented.§ First, it is noted that J_M (continuous or discrete) with the sets V and V_K satisfy the sufficient conditions of the minimax theorem and thus *Property 1*:

$$\min_{K \varepsilon V_K} \max_{v \varepsilon V} \bar{J}_M(K, v) = \max_{v \varepsilon V} \min_{K \varepsilon V_K} \bar{J}_M(K, v) \tag{37}$$

The value of K which minimizes $J_M(K, v)$ is the optimal Kalman filter gain, for that value of v. Therefore the minimax problem can be reduced to a maximization of $J_0(v)$ over the range of v, that is, *Property 2*:

$$\max_{v \varepsilon V}\left[\min_{K \varepsilon V_K} \bar{J}_M(K, v)\right] = \max_{v \varepsilon V} \bar{J}_0(v) \tag{38}$$

Further, since, *Property 3*: the optimal performance index $J_0(v)$ is concave in V, it is straightforward that *Property 4*: the maximum of $\bar{J}_0(v)$ is a global maximum.

In general the maximization indicated by Eq. (38) necessitates numerical techniques. These techniques are simplified somewhat by noting that *Property 5*: $\text{grad}_v \bar{J}_0(v)$ always exists. The above results and careful investigation of $\text{grad}_v \bar{J}_0(v)$ indicate that the S_1 filter is easily found. Diagonal elements of Q and R are set to their largest value. For off-diagonal elements steepest ascent techniques are appropriate. One need only insure that the definiteness requirements on Q and R are met at each step in the search.

Turning now to the minimax sensitivity filters, since

$$\min_{K \varepsilon V_K} \bar{S}(K, v) = 0,$$

it is clear that min-max does not equal max-min for the S^A and S^R sensitivity measures. However, the regions in V and V_K in which the minimax is found can be determined. First, though, some general properties are stated.

The following properties of the minimax sensitivity filters for both continuous and discrete applications will be stated again without proofs. *Property 6*: The absolute performance index $\bar{S}^A(K, v)$ is convex in V. *Property 7*: The relative performance index $\bar{S}^R(K, v)$ is convex in V. *Property 8*: The performance sensitivities are strictly convex in V_K for every $v \varepsilon V$.

A point x in a convex set X is called an extreme point of X, if there are no points $x_1, x_2 \varepsilon X$ such that $x = \alpha x_1 + (1 - \alpha)x_2$ for some α, $0 < \alpha < 1$, where $x_1 \neq x_2$. Further, if $f(x)$ is a convex scalar function of the variable x which is an element of a compact convex set X, then

$$\max_{x \varepsilon X} f(x) = \max_{x \varepsilon X_E} f(x) \tag{39}$$

where X_E is the set of extreme points of X. This important idea states that when looking for the maximum values of $\bar{S}(K, v)$ with respect to v, it is only necessary to consider the extreme points as stated in *Property 9*: Let V_E denote the extreme points of V, then

$$\min_{K \varepsilon V_K} \max_{v \varepsilon V} \bar{S}(K, v) = \min_{K \varepsilon V_K} \max_{v \varepsilon V_E} \bar{S}(K, v) \tag{40}$$

hence the minimax sensitivity filter is defined by that value of $\hat{K}$ such that

$$\max_{v \varepsilon V_E} \bar{S}(\hat{K}, v) = \min_{K \varepsilon V_K} \max_{v \varepsilon V_E} \bar{S}(K, v) \tag{41}$$

Hence *Property 10*: The value of $\hat{K}$ which defines the minimax filter is unique, and *Property 11*: The S_2 and S_3 filters for constant but uncertain Q and R are optimal for some $v \varepsilon V$, that is,

$$\hat{K} \varepsilon \mathscr{K}_0 \tag{42}$$

It is interesting to note at this point that the foregoing two properties are exactly those needed to justify the extensive

§ Results given in this section are also valid for time-varying Q and R; however, they are obtained most directly via variational techniques. See D'Appolito.[5]

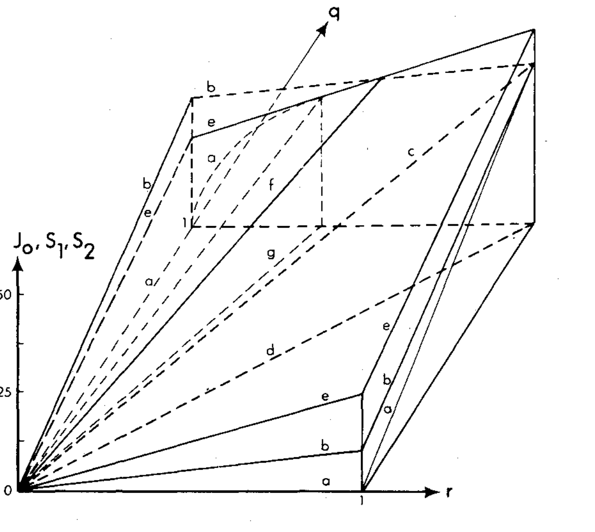

Fig. 1 S_1, S_2, and optimal error surfaces for example 1: a) J_0 (optimal) error surface; b) S_1 filter error plane; c) S_1, J_0 line of contact; d) projection of c on $q \times r$; e) S_2 filter error plane; f) S_2, J_0 line of contact; g) projection of f on $q \times r$.

sensitivity studies that have been used in practice for designing Kalman filters, in the presence of uncertainty.

The following property is due to Salmon,[8] but it has been modified somewhat to include the fact that the maximum value of S^A or S^R is attained at the extreme points of V. Thus, *Property 12*: Let $S(K, v)$ be any performance sensitivity (S^A or S^R) such that 1) $\hat{K}$ is an interior point of V_K and 2) $\partial S(K,v)/\partial K$ is continuous jointly with respect to K and v. Then the minimax value of S is attained at two or more distinct values of v, i.e., there exist $v_1, v_2 \varepsilon V_E$ (set of extreme points of V),

$$\min_{K \varepsilon V_K} \max_{v \varepsilon V_E} \bar{S}(K, v) = \bar{S}(\hat{K}, v_1) = \bar{S}(\hat{K}, v_2) \tag{43}$$

This property of the minimax sensitivity filters plays an important role in the algorithm developed to find the minimax gain.[9]

Using the properties outlined in this section systematic design of minimax filters is possible. Such techniques are discussed in Eqs. (5) and (9). The next section contains examples of such designs.

Examples

Example 1

This example illustrates the basic properties of minimax filters. Consider a first-order plant with noisy measurement

$$\dot{x}(t) = -x(t) + u(t); \quad z = x(t) + w(t) \tag{44}$$

where $\text{cov}[u(t)] = q\delta(t-\tau)$, $\text{cov}[w(t)] = r\delta(t-\tau)$, and q and r are assumed to be in the ranges

$$0 \leqq q \leqq 1, \quad 0 \leqq r \leqq 1 \tag{45}$$

Using the filter

$$\dot{\bar{x}} = -\bar{x} + k(z - \bar{x}); \quad k > -1 \tag{46}$$

J_0 and J_M are

$$J_0 = (r^2 + rq)^{1/2} - r, \quad J_M = [k^2 r + q]/[2(1+k)] \tag{47}$$

The minimax value of J_M occurs at $q = r = 1$ with $k = 0.414$. The minimax value of S^A is attained at the $q \times r$ extreme points (0, 1) and (1, 0) with $k = 1$. Since S^R is infinite when $r = 0$ or $q = 0$, the S_3 filter does not exist for this example.

A numerical comparison of the S_1 and S_2 filters is contained in Table 1. Notice that the S_1 filter provides a least upper bound on J_M at the expense of greater maximum deviation from optimality, whereas the opposite is true of the S_2 filter. The optimal S_1 and S_2 filter error surfaces are shown in Fig. 1.

Table 1 S_1, S_2 filter comparison for example 1

Extreme point q, r	J_0	S_1 J_M	S_1 $\Delta_1{}^a$	S_2 J_M	S_2 Δ_2
0, 0	0	0	0	0	0
0, 1	0	0.068	0.068	0.250	0.250
1, 0	0	0.353	0.353	0.250	0.250
1, 1	0.414	0.414	0	0.500	0.068
Max value	0.414	0.414	0.353	0.500	0.250

^{a} $\Delta \equiv J_M - J_0$.

Observe that the S_1 and S_2 filters are optimal for q/r ratios of 1 and 3, respectively.

Example 2

Consider the system

$$\dot{x}(t) = Fx(t) + u(t), \quad z_n = Hx_n + w_n \tag{48}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

$$H = [1, 0], \quad E[w_i w_j] = \delta_{ij}$$

The uncertain parameters set $V_Q(q_{11}, q_{22})$ is taken to be

$$1 \leqq q_{11} \leqq 4, \quad 4 \leqq q_{22} \leqq 9 \tag{49}$$

The infinite-time discrete filter is given by

$$\bar{x}_n = \Phi(\Delta t)\bar{x}_{n-1} + K(z_n - H\Phi(\Delta t)\bar{x}_{n-1}) \tag{50}$$

where

$$\Phi(\Delta t) = e^{F\Delta t}, \quad \Delta t = 0.1 \text{ sec}$$

The values of K for each minimax filter are

$$S_1: \quad K = \begin{bmatrix} 0.5316 \\ 0.6492 \end{bmatrix} \quad v = (4, 9)$$

$$S_2: \quad K = \begin{bmatrix} 0.4718 \\ 0.5840 \end{bmatrix} \quad v = (2.526, 6.458)$$

$$S_3: \quad K = \begin{bmatrix} 0.4723 \\ 0.5826 \end{bmatrix} \quad v = (2.541, 6.432)$$

The optimal, S_1, S_2, and S_3 filters for this example are compared in Table 2.

Example 3

In many instances the position error of a terrestrial inertial navigator is adequately described by a three-degree-of-freedom oscillator with a twenty-four hour period.[10] The three states of this oscillator, ψ_x, ψ_y, and ψ_z represent the small angular misalignment of the inertial platform about the computational co-

Table 2 Minimax filter comparison for example 2

Extreme point q_{11}	q_{22}	J_0	S_1 J_M	S_1 Δ_1	S_2 J_M	S_2 Δ_2	S_3 J_M	S_3 Δ_3
1	4	3.212	3.326	0.114	3.258	0.046	3.259	0.047
1	9	5.133	5.493	0.360	5.466	0.333	5.473	0.340
4	4	4.937	5.285	0.348	5.269	0.333	5.262	0.327
4	9	7.452	7.452	0	7.478	0.026	7.477	0.025
Max value		7.452	7.452	0.360	7.478	0.333	7.477	0.340

Table 3 Radial position error comparison for S_1 and S_2 navigation filters[a]

Uncertain parameter $q_{12}(10^{+5}), r_{12}$	J_0	S_1 J_M	S_1 Δ_1	S_2 J_M	S_2 Δ_2
0.3, 0.05	0.258	0.269	0.011	0.284	0.026
0.3, −0.05	0.221	0.269	0.048	0.247	0.026
−0.3, 0.05	0.262	0.269	0.007	0.283	0.021
−0.3, −0.05	0.220	0.269	0.049	0.246	0.026
0, 0	0.265	0.269	0.004	0.265	~0
Max value	0.269	0.269	0.049	0.284	0.026

[a] All data in nautical miles.

ordinate system. Inputs to the oscillator are the x, y, and z gyro random drift rates which are assumed to be first-order continuous Markov processes. The mean square values of the gyro drift rates are fairly well known. The x and y gyro drift rates are also known to be correlated, but the amount of correlation is uncertain. Continuous measurements of x and y position with uncertain crosscorrelation are available. The performance index of interest is the total rms radial position error

$$J_M = R_e[E\{\psi_x{}^2+\psi_y{}^2\}]^{1/2} \tag{51}$$

A complete specification of the position error problem is given in the Appendix. The sets V_Q and V_R for this problem are

$$V_Q = \begin{bmatrix} 0.4\times 10^{-5} & \pm 0.3\times 10^{-5} & 0 \\ \pm 0.3\times 10^{-5} & 0.4\times 10^{-5} & 0 \\ 0 & 0 & 0.18\times 10^{-5} \end{bmatrix} \tag{52}$$

$$V_R = \begin{bmatrix} 0.0625 & \pm 0.05 \\ \pm 0.05 & 0.0625 \end{bmatrix}$$

A minimax radial error of 0.269 nautical miles was attained at the point

$$q_{12} = -0.3\times 10^{-6}, \quad r_{12} = 0.234 \tag{53}$$

The S_2 filter for this example produced a minimax absolute radial deviation from optimum of 0.026 naut miles. The design point for this filter is

$$q_{12} = -0.15\times 10^{-6}, \quad r_{12} = -0.03 \tag{54}$$

The error performance of both filters is shown in Table 3. Notice that the S_1 filter error is insensitive to the value of the off-diagonal terms since it is designed for the point where the gradient with respect to these terms is zero.

Example 4

The system considered is a linear constant-coefficient tracking filter used in conjunction with the SAM-D guidance system.[11] There are three state variables: x_1 (position), x_2 (velocity) and x_3 (acceleration). It is assumed that the acceleration, x_3, is exponentially correlated with rms acceleration σ_a, so that

$$E[x_3(t_1)x_3(t_2)] = \sigma_a{}^2 e^{-|t_1-t_2|/\tau} \tag{55}$$

The differential equations of the target in vector form are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \tag{56}$$

where u is a white noise process whose spectral density magnitude can be derived from Eq. (55) to yield

$$E[u(t_1)u(t_2)] = (2\sigma_a{}^2/\tau)\delta(t_1-t_2) \tag{57}$$

The measurement z_n is the target position plus noise at discrete instances, i.e.,

$$z_n = Hx_n + w_n \tag{58}$$

where

$$H = [1 \ 0 \ 0] \tag{59}$$

and

$$E[w_n{}^2] = \sigma_m{}^2 \tag{60}$$

One of the difficulties encountered in the filtering of radar data is the fact that measurement accuracy is generally much better along the radar beam than across it. If the filtering is to be done in inertial coordinates, however, one may be constrained to use identical filters in all coordinates. Then a question arises as to what measurement accuracy the filters should be designed for, and therefore an alternative would be to build the S_1 filter for all coordinates. Consider a system in which

$$\Delta t = 0.5 \text{ sec}, \quad \tau = 5 \text{ sec}, \quad \sigma_a = 120 \text{ fps}$$

and σ_m is 30 ft along the radar beam and considerably greater across the beam. For example, a 2-mil radar will yield $\sigma_m = 300$ ft across the beam at a range of about 150,000 ft. Casting this problem into the framework of this paper yields

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5760 \end{bmatrix} \tag{61}$$

which is known, and

$$R = [\sigma_m{}^2] \tag{62}$$

where the range of $\sigma_m{}^2 = r_{11}$ is

$$900 \leqq r_{11} \leqq 90{,}000 \tag{63}$$

The S_1 filter gain is obtained by setting r_{11} to its maximum allowed value and computing the Kalman filter gain which is

$$K = \begin{bmatrix} 0.7954 \\ 1.1834 \\ 0.8765 \end{bmatrix} \tag{64}$$

The problem of finding the S_2 and S_3 filters for this example is quite simple. All one has to do is set the value of S (either S^A or S^R) at the extreme point $\sigma_m{}^2 = 900$ equal to S at the other extreme point $\sigma_m{}^2 = 90{,}000$, and solve for the R which satisfies this condition. The minimax gain is just the Kalman filter gain for that value of R which satisfies the constraint equation.

Table 4 summarizes the results for this example. All the quantities are rms values. The quantities D_2 and D_3 are the rms value of the difference between J_M and J_0 for the S_2 and S_3 filters, respectively. The S_2 minimax sensitivity filter is defined for

$$\sigma_m{}^2 = 27097.3 \tag{65}$$

and

$$K = \begin{bmatrix} 0.575 \\ 0.483 \\ 0.180 \end{bmatrix} \tag{66}$$

The S_3 filter is defined for

$$\sigma_m{}^2 = 9995.0 \tag{67}$$

and

Table 4 S_2 and S_3 filter performance

σ_m	J_0	S_2 J_M	S_2 D_2	S_3 J_M	S_3 D_3
300.0	310.16	328.74	108.95	372.03	205.43
200.0	245.49	247.12	28.31	266.05	102.55
100.0	173.80	181.29	51.56	173.80	0
80.0	157.84	171.70	67.59	159.18	20.62
30.0	111.54	155.92	108.95	133.79	73.88

$$K = \begin{bmatrix} 0.641 \\ 0.642 \\ 0.290 \end{bmatrix} \tag{68}$$

As a comparison, the results for the S_1 and S_2 filters are plotted in Fig. 2. The vertical axis is the rms value of the total error in the estimate. Note the degradation in performance of the S_1 filter when the measurement accuracy is 30 ft. This clearly illustrates the shortcomings of the S_1 filter in attempting to design for the worst case when the range of the unknown parameter is very large. In comparison the S_2 filter, although having a larger maximum error, still performs well for small measurement errors.

Conclusion

A minimax approach to the design of linear filters for state estimation when large uncertainties in plant and measurement noise covariances are present has been given. This approach results in a unique fixed filter design which places a least upper bound on a given sensitivity measure over the assumed range of uncertain parameters. The resulting filters are identical in form to the Kalman filter, provide nearly optimal performance over the entire range of uncertain statistics, and are independent of the actual noise statistics. The design approach, therefore, circumvents one of the major limitations on the use of the Kalman filter, namely, the need to have exact knowledge of plant and measurement noise statistics.

Appendix

Navigator Error State Equations

$$\begin{bmatrix} \dot\psi_x \\ \dot\psi_y \\ \dot\psi_z \\ \dot\varepsilon_x \\ \dot\varepsilon_y \\ \dot\varepsilon_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_v & 0 & K_1 & 0 & 0 \\ \Omega_v & 0 & \Omega_H & 0 & K_1 & 0 \\ 0 & -\Omega_H & 0 & 0 & 0 & K_1 \\ 0 & 0 & 0 & -\beta_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_y & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_z \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Measurement Equations

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & R_e & 0 & 0 & 0 & 0 \\ -R_e & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \\ \vdots \\ \varepsilon_z \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where

ψ_x, ψ_y, ψ_z = platform misalignment (rad)

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = gyro drift rates (deg/hr)

$\Omega_V = \Omega_H = 0.186$ rad/hr

$K_1 = 0.01734$ rad/deg

$R_e = 3437$ naut miles

$\sigma_{\varepsilon_x} = \sigma_{\varepsilon_y} = 0.0014$ deg/hr

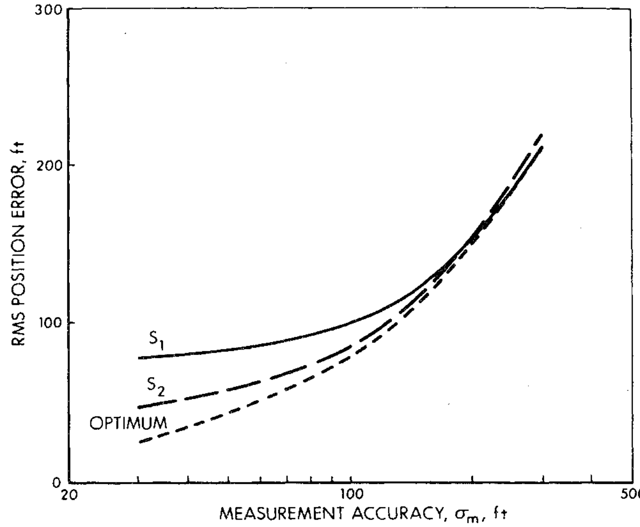

Fig. 2 Error performance of the S_1 and S_2 filters for SAM-D guidance system.

$\beta_x = \beta_y = 1$/hr

$\sigma_{\varepsilon_z} = 0.003$ deg/hr

$\beta_z = 0.1$/hr

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